Jackson
3.4 (b)

$$\Phi = \sum_{l,m} \left[A_{lm} r^{l} + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta, \phi)$$

$$\langle Y_{l'm'} | Y_{lm} \rangle = \delta_{l'l}$$

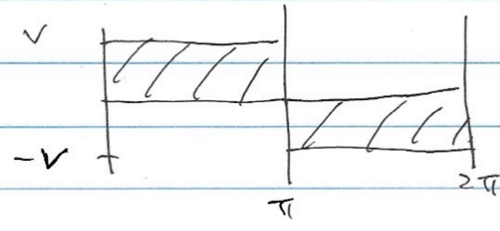
$$\Phi = \sum_{lm} A_{lm} r^{l} Y_{lm}(\theta, \phi)$$

$$\langle Y_{lm} | \Phi \rangle \Big|_{r=a} = A_{lm} a^{l}$$

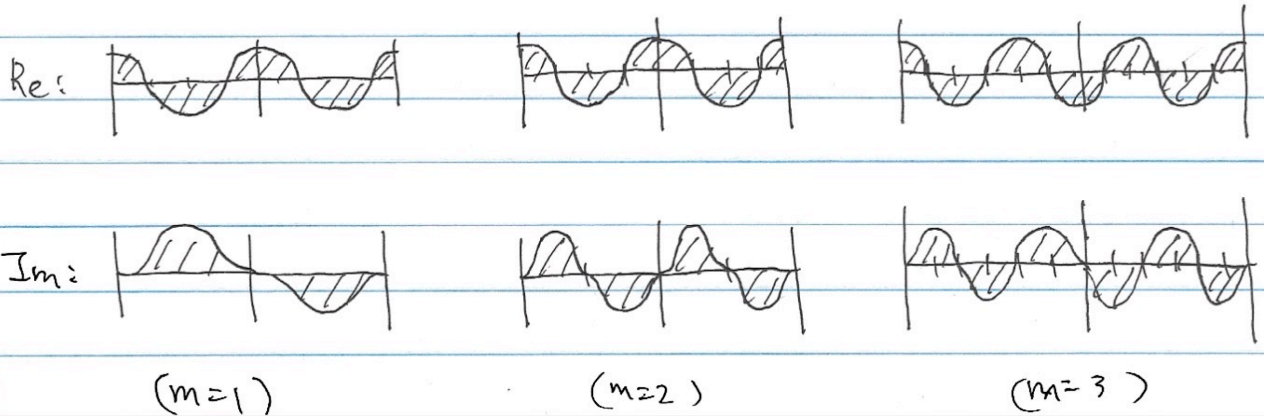
$$A_{lm} = a^{-l} \langle Y_{lm} | \Phi \rangle \Big|_{r=a}$$

$$= a^{-l} \int_0^{2\pi} d\phi \int_{0}^{\pi} \sin\theta \, d\theta \; Y^{*}_{lm}(\theta, \phi) \, \Phi(a, \theta, \phi)$$

When $n=1$, the ϕ dependence of Φ is



$e^{im\phi}$ looks like



$$\int e^{-i\phi}\,\Phi\,d\phi = -4Vi\,, \quad \int e^{i\phi}\,\Phi\,d\phi = 4Vi$$

$$\int e^{-2i\phi}\,\Phi\,d\phi = 0$$

$$\int e^{-3i\phi}\,\Phi\,d\phi = -\frac{4}{3}Vi\,, \quad \int e^{3i\phi}\,\Phi\,d\phi = \frac{4}{3}Vi$$

$$A_{11} = a^{-1}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\; Y_{11}^{*}(\theta,\phi)\,\Phi(a,\theta,\phi)$$

$$= a^{-1}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\left[-\sqrt{\frac{3}{8\pi}}\sin\theta\right]e^{-i\phi}\,\Phi$$

$$= -a^{-1}\sqrt{\frac{3}{8\pi}}\int_0^{\pi}\sin^2\theta\,d\theta\;[-4Vi]$$

$$= \frac{2\cancel{4}Vi}{a}\sqrt{\frac{3}{8\pi}}\,\frac{\pi}{\cancel{2}} = \boxed{\frac{2\pi V}{a}\sqrt{\frac{3}{8\pi}}\,i}$$

$$A_{1-1} = a^{-1}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\left[\sqrt{\frac{3}{8\pi}}\sin\theta\right]e^{i\phi}\,\Phi$$

$$= \boxed{\frac{2\pi V}{a}\sqrt{\frac{3}{8\pi}}\,i}$$

$$A_{21} = a^{-2} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \, Y^*_{21}(\theta,\phi)\, \Phi$$

$$= a^{-2} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \left[-\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta \right] e^{-i\phi}\, \Phi$$

$$= a^{-2} \sqrt{\frac{15}{8\pi}}\, 2V_i \int_0^{\pi} \sin^2\theta \cos\theta \, d\theta$$

$$= 0$$

$$\Rightarrow A_{2-1} = 0.$$

$$A_{31} = a^{-3} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \left[-\frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin\theta \left(5\cos^2\theta - 1\right) \right] e^{-i\phi}\, \Phi$$

$$= \frac{a^{-3}}{\cancel{2}\cancel{4}} \sqrt{\frac{21}{4\pi}}\, \cancel{2} V_i \int_0^{\pi} \sin^2\theta d\theta \left[5\cos^2\theta - 1 \right]$$

$$= \sqrt{\frac{21}{4\pi}}\, \frac{1}{\cancel{2}a^3}\, V_i \, \frac{\pi}{8}$$

$$= \sqrt{\frac{21}{4\pi}}\, \frac{1}{a^3}\, \frac{1}{\cancel{16}8}\, \pi V_i$$

Similarly, $A_{3-1} = \boxed{\sqrt{\frac{21}{4\pi}}\, \frac{\pi V_i}{\cancel{16}8a^3}}$

$$A_{33} = a^{-3} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, Y_{33}^{*}(\theta, \phi) \, \Phi$$

$$= a^{-3} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \left[-\frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3\theta \, e^{-3i\phi} \right] \Phi$$

$$= -a^{-3} \frac{1}{4} \sqrt{\frac{35}{4\pi}} \int_0^{\pi} \sin^4\theta \, d\theta \int_0^{2\pi} d\phi \, e^{-3i\phi} \, \Phi$$

$$= \frac{a^{-3}}{\cancel{4}} \sqrt{\frac{35}{4\pi}} \left[\frac{\cancel{4}}{3} V i \right] \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= \frac{a^{-3}}{\cancel{2}\cancel{4}} \sqrt{\frac{35}{4\pi}} \frac{\cancel{2}}{\cancel{3}} V i \frac{\cancel{3}\pi}{8}$$

$$= \boxed{\frac{\pi V}{\cancel{16}8 a^3} \sqrt{\frac{35}{4\pi}} \, i}$$

$$A_{3-3} = \frac{\pi V}{8\cancel{16} a^3} \sqrt{\frac{35}{4\pi}} \, i \quad \text{similarily.}$$

$$\Phi = \frac{2\pi V}{1}\left(\frac{r}{a}\right)\sqrt{\frac{3}{8\pi}}\, i\left[Y_{11} + Y_{1-1}\right]$$

$$+ \frac{\pi V}{8}\left(\frac{r}{a}\right)^3\sqrt{\frac{21}{4\pi}}\, i\left[Y_{31} + Y_{3-1}\right]$$

$$+ \frac{\pi V}{8}\left(\frac{r}{a}\right)^3\sqrt{\frac{35}{4\pi}}\, i\left[Y_{33} + Y_{3-3}\right]$$

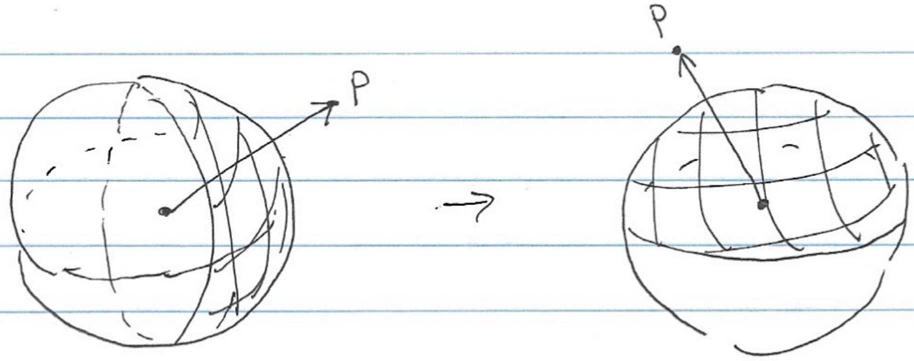
$$+ \cdots$$

$$Y_{11} + Y_{1-1} = -\sqrt{\frac{3}{8\pi}}\sin\theta\left[e^{i\phi} - e^{-i\phi}\right]$$

$$Y_{31} + Y_{3-1} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta\,(5\cos^2\theta - 1)\left[e^{i\phi} - e^{-i\phi}\right]$$

$$Y_{33} + Y_{3-3} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta\left[e^{3i\phi} - e^{-3i\phi}\right].$$

$$\Rightarrow \boxed{\Phi = V\left[\frac{3}{2}\left(\frac{r}{a}\right)\sin\theta\sin\phi + \frac{21}{64}\left(\frac{r}{a}\right)^3\sin\theta\left[5\cos^2\theta - 1\right]\sin\phi + \frac{35}{64}\left(\frac{r}{a}\right)^3\sin^3\theta\,\sin 3\phi\right] + \cdots}$$



The transition is made by setting ϕ fixed at $\frac{\pi}{2}$, and $\theta \to \frac{\pi}{2} - \theta$. This makes $\sin\phi \to \pm 1$, $\cos\theta \to \sin\theta$, $\sin\theta \to \cos\theta$

$$\Rightarrow \Phi \to V\left[\frac{3}{2}\left(\frac{r}{a}\right)\cos\theta + \frac{21}{64}\left(\frac{r}{a}\right)^3 \cos\theta\left[5\sin^2\theta - 1\right] - \frac{35}{64}\left(\frac{r}{a}\right)^3\cos^3\theta\right] + \cdots$$

The $\left(\frac{r}{a}\right)^3$ order terms can be expanded to give

$$\left(\frac{r}{a}\right)^3\left(\frac{1}{64}\right)\left[(21)\cos\theta\left[5\sin^2\theta - 1\right] - 35\cos^3\theta\right]$$

$$\Downarrow$$

$$105\cos\theta\sin^2\theta - 21\cos\theta - 35\cos^3\theta$$
$$= 105\cos\theta - 105\cos^3\theta - 21\cos\theta - 35\cos^3\theta$$
$$= 84\cos\theta - 140\cos^3\theta$$
$$= 28\left(3\cos\theta - 5\cos^3\theta\right)$$
$$= \boxed{-56 P_3[\cos\theta]}$$

Recall that $P_1[\cos\theta] = \cos\theta$. Putting everything together

$$\Phi = V\left[\frac{3}{2}\left(\frac{r}{a}\right) P_1[\cos\theta] - \frac{56}{64}\left(\frac{r}{a}\right)^3 P_3[\cos\theta] + \cdots\right]$$

$$= V\left[\frac{3}{2}\left(\frac{r}{a}\right) P_1[\cos\theta] - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3[\cos\theta] + \cdots\right]$$

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